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The Grassmann Euclidean group Sp(2) ∧ T(2) and its representations

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Abstract. Extended BRST supersymmetry, wherein the ghosts are treated equally, can be viewed as the Grassmann inhomogeneous rotation group in two dimensions, Sp(2) ∧ T(2). The representations are labelled by pseudomass and pseudospin, and the physical state vectors correspond to wavepackets over the fermionic momentum. Irreducible field representations are constructed.

1. The BRST superalgebra

If one treats (Bonora *et al* 1981) the fictitious ghost and auxiliary fields that accompany a quantised gauge field, in an arbitrary covariant gauge, on an equal footing (Delbourgo and Jarvis 1982), one discovers that they transform into one another according to the extended BRST superalgebra:

$$\begin{aligned}
&[J_{kl}, J_{mn}] = \mathrm{i}(\eta_{km}J_{ln} + \eta_{kn}J_{lm} + \eta_{lm}J_{kn} + \eta_{ln}J_{km})\\
&[J_{kl}, P_m] = \mathrm{i}(\eta_{km}P_l + \eta_{lm}P_k) \qquad\qquad (1)\\
&\{P_m, P_n\} = 0.
\end{aligned}$$

The corresponding supergroup, Sp(2) ∧ T(2), can be regarded as the Grassmann version of the Euclidean group in two dimensions. In the algebra (1) the indices are two-valued (say 1, 2) because they encompass the BRST and the dual transformations. The P generate the supertranslations while the J (*symmetric* in their indices) rotate and scale the ghosts; see § 3. We adopt the consistent reality and finite-dimensionality assignments:

$$P_1^\dagger = P_2 \qquad \text{and} \qquad J_{11}^\dagger = J_{22} \qquad J_{12}^\dagger = -J_{12}$$

above, with the metric $\eta_{12} = 1$ used to raise and lower indices according to the rules

$$P_m = \eta_{mn}P^n \qquad \eta^{lm}\eta_{mn} = \delta^l{}_n \qquad \text{etc.}$$

Since this is the fermionic analogue of an inhomogeneous rotation group, we may anticipate that its representations will bear certain resemblances to the familiar Poincaré group representations. That is indeed the case, in that analogues of mass and spin can be used to characterise the eigenstates of the superalgebra. However, when the 'momentum' is used to complete the labelling of eigenvectors, the 'states' are unusual because they yield a-number eigenvalues for the fermionic operators and, if adopted as they stand, can give rise to nilpotent eigenvalues of observables, which is physically nonsensical. We show in the appendix, with reference to the Grassmann oscillator, that the correct way to circumvent this problem is to define 'wavepackets' over such idealised Grassmann states; the procedure is no different in principle to what one is accustomed in ordinary wave mechanics for obtaining normalisable physical wavefunctions as superpositions of monochromatic functions. Given that the Dirac notation can be consistently generalised to incorporate Grassmann operators and states (see the appendix for fuller confirmation), we investigate in this section how to set up and label the eigenvectors of the extended BRST superalgebra. In the following section we will show how to construct the irreducible fields for these representations and in the final section we shall compare our work with other treatments of the BRST group especially for the case of translationally trivial (i.e. 'physical') states, $\boldsymbol{p}=0$.

The Casimirs which label an irreducible representation have to be scalar and supertranslation invariant. Clearly, the (nilpotent) squared pseudomass

$$p^2 = P_k P^k/2 = P_2 P_1 \tag{2}$$

will do, whereas $J_{mn}J^{mn}$ is unacceptable. However, we may define a pseudospin operator, the analogue of the Pauli-Lubanski vector for the Poincaré group, namely

$$W_{klm} = P_k J_{lm} + P_l J_{mk} + P_m J_{kl} = J_{lm} P_k + J_{mk} P_l + J_{kl} P_m. \tag{3}$$

It is translationally invariant,

$$\{W_{klm}, P_n\} = 0 \tag{4}$$

and closes on itself as well as the 'momentum',

$$\{W_{klm}, W_{pqr}\} = \mathrm{i}(P_p \eta_{qk} + P_q \eta_{pk}) W_{rlm} + \text{cyclic permutations in } p,\ q,\ r \text{ and } k,\ l,\ m \tag{5}$$

just like the Pauli-Lubanski vector operator. Obviously the squared pseudospin

$$w^2 = W_{klm} W^{klm}/48 \tag{6}$$

can be taken as the second Casimir (nilpotent) operator of $\mathrm{Sp}(2) \wedge \mathrm{T}(2)$. Because it involves the square of P, each component of which is nilpotent, it is straightforward to prove that

$$W_{klm} W^{klm} = 6 P_k P^k J^{mn} J_{mn} \tag{6'}$$

and as we shall presently see, the actual spin j of an irreducible finite-dimensional representation is determined through the relation,

$$w^2 = p^2 j(j+1). \tag{6''}$$

The case $p^2 = 0$ is rather special and is postponed to § 3.

The only remaining question is how we are to label the rest of the basis vector, i.e. find the remaining operators of the 'complete (anti)commuting set'. There are at least two possibilities.

(A) The first is the straight analogue of the Poincaré states $|j; p\rangle$. It is to use the eigenstates $|j; p\}$ of the full Grassmann momentum P. The p_k are a numbers now and the $|\}$ signifies that the representation is non-unitary. Of course, in this basis the eigenvalue of $P_k P^k/2$ is simply $p_2 p_1$. Since a 'rotation' of the momentum causes the change

$$\exp(\mathrm{i}\alpha J/2) P_m \exp(-\mathrm{i}\alpha J/2) = (\exp \alpha)_m^{\ k} P_k$$

one readily finds that a finite supertransformation of these states is described by

$$\exp(-\mathrm{i}\alpha J/2) \exp(-\mathrm{i}\xi P)|j; p\} = \exp(-\mathrm{i}\xi p)|j; \mathrm{e}^{\alpha} p\} \tag{7}$$

where α^{mn} are three rotation parameters (c numbers) and ξ_k are two translation parameters (a numbers).

(B) A second basis is obtained by using a helicity-like variable λ in place of the 'direction of P'. For convenience we take λ to be related to the eigenvalue of J_{12}. In this connection let us momentarily digress and record the isomorphism between Sp(2) and the usual rotation group SU(2) via the identification of the generators:

$$J_{12} \leftrightarrow -2\mathrm{i}J_3 \qquad J_{11} \leftrightarrow 2(J_1 + \mathrm{i}J_2) \qquad J_{22} \leftrightarrow 2(J_1 - \mathrm{i}J_2). \tag{8}$$

This means that an irreducible finite-dimensional Sp(2) representation can be expressed in terms of a $2j$ symmetric multispinor A (with $j = 0, \frac{1}{2}, 1, \ldots$) upon which the Sp(2) generators act as follows:

$$J_{kl}A_{m_1m_2\ldots m_{2j}} = \mathrm{i}[\eta_{km_1}A_{lm_2\ldots m_{2j}} + \eta_{lm_1}A_{km_2\ldots m_{2j}} + \eta_{km_2}A_{m_1l\ldots m_{2j}} + \eta_{lm_2}A_{m_1k\ldots m_{2j}} + \ldots + \eta_{km_{2j}}A_{m_1m_2\ \ l} + \eta_{lm_{2j}}A_{m_1m_2\ldots k}]. \tag{9}$$

In this construction one perceives that $A_{11\ldots1}$ is the highest and $A_{22\ldots2}$ is the lowest weight of the representation possessing, respectively, the J_{12} eigenvalues $-2\mathrm{i}j$ and $2\mathrm{i}j$, thereby agreeing with (7). Indeed the general A spinor carrying λ_1 indices of type 1 and λ_2 indices of type 2, with $\lambda_1 + \lambda_2 = 2j$, has J_{12} eigenvalue $-\mathrm{i}(\lambda_1 - \lambda_2)$.

In this basis then we write our states as $|p^2j; \lambda\}$ and take

$$J_{12}|p^2j; \lambda\} = -2\mathrm{i}\lambda|p^2j; \lambda\}. \tag{10}$$

It is easy to see that a finite Sp(2) rotation is obtained from the normal SU(2) representation $\mathrm{D}(\boldsymbol{\alpha})$ by identifying the rotation parameters

$$\alpha_1 = \alpha^{11} + \alpha^{22} \qquad \alpha_2 = \mathrm{i}(\alpha^{11} - \alpha^{22}) \qquad \alpha_3 = -2\mathrm{i}\alpha^{12} \tag{11}$$

and making the appropriate continuation. Because P_1 raises λ by $\frac{1}{2}$ and P_2 lowers λ by $\frac{1}{2}$, we may define the action of the P on these 'helicity' states to be

$$P_1|\lambda\} = p_1|\lambda + \tfrac{1}{2}\} \qquad \text{and} \qquad P_2|\lambda\} = p_2|\lambda - \tfrac{1}{2}\}$$

so as to obtain the correct eigenvalue for the Casimir P_kP^k. In this manner the action of all the operators is *totally* fixed.

2. Field representations

In gauge theories, it is more conventional to consider functions $\Phi(\theta)$ as representations of Sp(2)∧T(2), where θ is a Grassmann coordinate space which is appended to spacetime. This is the spirit of superspace and superfields. The generators of the algebra act as differential operators on these functions:

$$\begin{aligned} P_k\Phi &= \mathrm{i}\partial\Phi/\partial\theta^k = \mathrm{i}\partial_k\Phi \\ J_{kl}\Phi &= [\mathrm{i}(\theta_k\partial_l + \theta_l\partial_k) + \Sigma_{kl}]\Phi. \end{aligned} \tag{12}$$

In (12) Σ acts as a Sp(2) spin operator in the same way that J acts on multispinors in (9). It is then relatively easy to set up the eigenfunctions corresponding to the choices of basis (A) and (B) of the previous section.

(A) Because we are dealing with P eigenfunctions, it is perfectly obvious that Φ must have the form

$$\Phi(\theta) = A\exp(\mathrm{i}\theta_kp^k)$$

where A is independent of θ. If we then want a 'spin j' representation, we must clearly choose A to be a multispinor of type (9). It is then easy to see that

$$W_{klm}W^{klm}\Phi = 48j(j+1)p_k p^k \Phi$$

as anticipated earlier. This finally means that the representation functions in this particular basis are

$$\{m_1 \ldots m_{2j}, \theta \,|\, j, p\} = \Phi_{m_1 \ldots m_{2j}}(\theta) = A_{m_1 \ldots m_{2j}} \exp(\mathrm{i}\theta p) \tag{13}$$

(B) Instead, using eigenfunctions of J_{12}, we find that the expansion coefficients of Φ:

$$\Phi(\theta) = a + \theta_k b^k + \theta_k \theta^k c/2 \tag{14}$$

must obey

$$p^2 a = c \qquad p^2 b = 0$$

and

$$\Sigma_{12} a = -2\mathrm{i}\lambda a \qquad \Sigma_{12} b_1 = -2\mathrm{i}(\lambda + \tfrac{1}{2}) b_1 \qquad \Sigma_{12} b_2 = -2\mathrm{i}(\lambda - \tfrac{1}{2}) b_2 .$$

Consequently the expansion coefficients must be taken to equal

$$a = A_{11\ldots 22} \qquad b_k = (p_k A_{11\ldots 22} + \text{symmetrisation over } k) \qquad c = p^2 a$$

where the number of 1 and 2 indices on the multispinor A yield 'helicity' λ. It is then clear that (6″) will automatically be satisfied (the b and c components become irrelevant). Thus, for this basis,

$$\begin{aligned}\Phi_{11\ldots 22}(\theta) &= \{11 \ldots 22\theta \,|\, p^2 j, \lambda\} \\ &= (1 + p^2 \theta_k \theta^k) A_{11\ldots 22} + \theta^k (p_k A_{11\ldots 22} + \text{symmetric permutations}).\end{aligned}$$

This completes our discussion of the field description. We must now compare our work with other treatments of the extended BRST group representations, focusing our attention on the limit $p_k \to 0$ where the states become 'physical'.

3. Physical and null states

Here we shall establish the connection between the algebra (1) and earlier important references to this subject (Kugo and Ojima 1979, Nakanishi and Ojima 1980, Bonora *et al* 1981, Nishijima 1984). The conventional treatment introduces the ghost charge Q_c and the generators Q_B and $\bar{Q}_B$ of BRST and $\overline{\text{BRST}}$ transformations. The transcription to our notation is

$$Q_c \leftrightarrow J_{12} \qquad Q_B \leftrightarrow P_1 \qquad \bar{Q}_B \leftrightarrow P_2 .$$

This subset generates a $U(1) \wedge T(2)$ subalgebra. However, soon after the discovery of this limited subalgebra, it was recognised that gauge theory (Nakanishi and Ojima 1980) and gravity in the 'Landau' gauge admitted a large choral symmetry group (Nakanishi 1982) which included the two generators, $Q = Q(c, c)$ and $\bar{Q} = Q(\bar{c}, \bar{c})$, among many others. These satisfied the extra commutation rules

$$\begin{aligned}&\mathrm{i}[Q_c, Q] = 2Q \qquad \mathrm{i}[Q_c, \bar{Q}] = -2\bar{Q} \\ &[Q, \bar{Q}] = 4\mathrm{i}Q_c \qquad [Q, \bar{Q}_B] = 2\mathrm{i}Q_B \qquad [\bar{Q}, Q_B] = -2\mathrm{i}\bar{Q}_B .\end{aligned} \tag{15}$$

Our own treatment (Delbourgo and Jarvis 1982), which carries over to *any* gauge, lets us identify the remaining Sp(2) generators as

$$Q \leftrightarrow J_{11} \qquad \bar{Q} \leftrightarrow J_{22}$$

and allows them *both to be conserved.* Thus we have regained the starting superalgebra (1). In our version, with the reality properties stated in § 1, the 'ghost number' or eigenvalue of J_{12}, corresponds to $-2\mathrm{i}\lambda$, where λ is the helicity label.

The 'charge' operators above can be expressed as time integrals over Noether currents in the standard manner. Particularly relevant are the generators of Sp(2) transformations:

$$J_{ij} = \mathrm{i} \int \omega_i D_\mu \omega_j \, \mathrm{d}\sigma^\mu \tag{16}$$

where ω is the Sp(2) ghost doublet. Equally important is the fact that one can write (Bonora *et al* 1981)

$$J_{ij} = \{P_i, R_j\} + \{P_j, R_i\} \tag{17}$$

where

$$R_j = \int \omega_j A_\mu \, \mathrm{d}\sigma^\mu \tag{18}$$

because of the equal-time commutators.

The representations of U(1) $\wedge$ T(2) were exhaustively studied in the early 1980s. Greater strides were taken by Nishijima who considered the enlarged algebra SU(2) $\wedge$ T(2), although in his case the charges J_{11} and J_{22} were not conserved except in the Landau gauge. Nevertheless he was able to identify three types of irreducible representation: singlets, quartets and infinite chains; he excluded the last case as being infinite dimensional and thus unrealistic and we too have done the same by confining our Sp(2) basis to a finite-dimensional one (whence the reality conditions on the J_{ij}). Nishijima's quartet states simply correspond to the basis vectors stated in 1(B) and 2(B), but of course our own study has stressed the importance of the Casimirs of the extended group and the usefulness of Grassman vectors.

It remains to say a few words about the situation when the supertranslation group is trivially represented, $P \to 0$, since both the Casimirs p^2 and w^2 of (2) and (6) disappear. This case is important because 'physical' states must be BRST invariant (like the quantum action) at the very least. In that circumstance we should focus on the Sp(2) group and its Casimir, which roughly speaking is the ratio of w^2 to p^2 and *non-zero in general.* If it happens that the ghost charge, or eigenvalue of J_{12}, does *not* vanish then it is easy to show that

$$\begin{aligned} J_{12}|j\lambda\rangle &= -2\mathrm{i}\lambda|j\lambda\rangle = (\{P_1, R_2\} + \{P_2, R_1\})|j\lambda\rangle \\ &= P_1|\chi_2\rangle + P_2|\chi_1\rangle \end{aligned} \tag{19}$$

where

$$|\chi_i\rangle = R_i|j\lambda\rangle \tag{19'}$$

proving that $|j\lambda\rangle$ is 'null' in the standard parlance. This is one of the principal theorems in this BRST area; there are a few corollaries (Bonora *et al*) and we have nothing further to add on that score. Remember that the genuine physical states must also carry *zero ghost charge*, i.e. have $\lambda = 0$, and these are the ones that correspond to BRST singlets.

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Appendix. Grassmann states and the Grassmann oscillator

The one-dimensional fermionic oscillator is defined by the algebra, $\{Q, Q^\dagger\} = 1$, where $Q^\dagger$ creates one unit fermion number and Q destroys it. In the Bose case, we know that we can construct (coherent) states which are eigenfunctions of the destruction operator; let us analogously define the Grassmann states

$$|q\rangle = \exp(Q^\dagger q)|0\rangle = |0\rangle - q|1\rangle \tag{A1}$$

where $|0\rangle$ is the ground state (annihilated by Q) and q is an a number. This is to ensure that

$$Q|q\rangle = q|q\rangle.$$

The conjugate relations are

$$\langle q| = \langle 0| \exp(q^* Q) = \langle 0| + q^*\langle 1| \qquad \langle q|Q^\dagger = \langle q|q^*.$$

Also it is simple to establish the completeness relation:

$$1 = |0\rangle\langle 0| + |1\rangle\langle 1| = \int \mathrm{d}q^*\, \mathrm{d}q\, |q\rangle \exp(-q^* q)\langle q| \tag{A2}$$

as a Berezinian integral over Grassmann states, in complete analogy to the Bose case.

If we then define an observable (something like a Hamiltonian) as the operator $H = Q^\dagger Q$ and take the Grassmann state expectation value

$$\langle q|H|q\rangle = q^* q,$$

we seem to be led into an absurdity, since the result is a nilpotent real number! This is quite unacceptable for a physical operator. However this is no cause for despair and the rejection of such Grassmann states. We are familiar with such occurrences in ordinary wave mechanics; pure eigenstates of momentum are not permissible either (though we can get round them by coping with delta functions and other distributions) because they are not normalisable. Strictly, we should deal with momentum wavepackets centred about some average value. Let us adopt the same attitude for the Grassmann oscillator states $|q\rangle$ and look for superpositions over them to see if they will prove physically acceptable too.

A good example is provided by the standard (normalised) coherent state

$$|c\rangle = (|0\rangle + c|1\rangle)/(1+|c|^2)^{1/2}$$

with its sensible expectation value

$$\langle c|H|c\rangle = |c|^2(1+|c|^2)^{-1}$$

because c is an ordinary c number. In fact it is easily shown that the overlap of this physical state with a Grassmann state must equal

$$\langle q \mid c\rangle = (1 + q^* c)/(1+|c|^2)^{1/2} = \mathrm{e}^{q^* c}/(1+|c|^2)^{1/2}. \tag{A3}$$

It is perfectly acceptable to use this kind of superposition over the unphysical $|q\rangle$ to create the physical states of the theory. Providing one is happy to include wavefunctions that are polynomials in Grassmann variables, such as (A3), and adopts the normal conventions of Grassmann integration like (A2), there are no signs of any inconsistencies anywhere in such an extension of the Dirac formalism.

All this becomes more convincing if we generalise to the N-dimensional Grassmann oscillator. A recent paper by Finkelstein and Villasante (1985) makes extensive use of Grassmann variables in order to obtain the wavefunctions and Green functions as analogues of the usual bosonic Hermite representation; it forms a helpful supplement to this appendix. Here we have a set of N annihilation and creation operators, obeying

$$\{Q_i, Q_j^\dagger\} = \delta_i^j. \tag{A4}$$

Once again, since all the Q_i anticommute with one another, we construct the Grassmann eigenfunctions

$$|q\rangle = \exp(Q_i^\dagger q_i)|0\rangle = |0\rangle - q_i|i\rangle + \tfrac{1}{2}q_i q_j|i,j\rangle - \ldots$$

$$|i,j\rangle = Q_j^\dagger Q_i^\dagger|0\rangle$$

etc, to be eigenstates of the various annihilation operators:

$$Q_i|q\rangle = q_i|q\rangle$$

where the q_i are a numbers. The completeness relation is expressible as a Berezinian integral over the Grassmann states:

$$1 = |0\rangle\langle 0| + |i\rangle\langle i| + \tfrac{1}{2}|i,j\rangle\langle i,j| + \ldots$$

$$= \int \mathrm{d}^N q^* \, \mathrm{d}^N q \; |q\rangle \exp(-q_r^* q_r)\langle q|.$$

Any physical state should then be written as a superposition over the $|q\rangle$; for instance, an ordinary coherent state can be written as an integral over

$$\langle q | c\rangle = \exp(q_i^* c_i)$$

multiplied into the $|q\rangle$, even in the N-dimensional problem, where c is a c number. Or, more generally, allowing for arbitrary Fock wavefunctions coefficients c, we can envisage the general Grassmann packet

$$\langle q | c\rangle = 1 + q_i^* c_i + \tfrac{1}{2} q_i^* q_j^* c_{ij} + \ldots.$$

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